

Structure of the largest idempotent-product free sequences in semigroups

Guoqing Wang

Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, P. R. China

Email: gqwang1979@aliyun.com

Abstract

Let S be a finite semigroup, and let $E(S)$ be the set of all idempotents of S . Gillam, Hall and Williams proved in 1972 that every S -valued sequence T of length at least $|S| - |E(S)| + 1$ is not (strongly) idempotent-product free, in the sense that it contains a nonempty subsequence the product of whose terms, in their natural order in T , is an idempotent, which affirmed a question of Erdős. They also showed that the value $|S| - |E(S)| + 1$ is best possible.

Here, motivated by Gillam, Hall and Williams' work, we determine the structure of the idempotent-product free sequences of length $|S \setminus E(S)|$ when the semigroup S (not necessarily finite) satisfies $|S \setminus E(S)|$ is finite, and we introduce a couple of structural constants for semigroups that reduce to the classical Davenport constant in the case of finite abelian groups.

Key Words: *Idempotent-product free sequences; Erdős-Burgess constant; Davenport constant; Zero-sum*

1 Introduction

Let S be a nonempty semigroup, endowed with a binary associative operation $*$ on S , and denote by $E(S)$ the set of idempotents of S , where $x \in S$ is said to be an idempotent (in S) if $x * x = x$. Our interest in semigroups and idempotents comes from the following question of P. Erdős to D.A. Burgess [2]:

If S is a finite nonempty semigroup of order n , does any S -valued sequence T of length n contain a nonempty subsequence the product of whose terms, in any order, is an idempotent?

In 1969, Burgess [2] gave an answer to this question in the case that S is commutative or contains only one idempotent. Shortly after, this question was completely affirmed by D.W.H. Gillam, T.E. Hall and N.H. Williams, who actually proved the following stronger result:

Theorem A. ([8]) *Let S be a finite nonempty semigroup. Any S -valued sequence of length $|S| - |E(S)| + 1$ contains one or more terms whose product (in their natural order in this sequence) is an idempotent; In addition, the bound $|S| - |E(S)| + 1$ is optimal.*

That better bounds can be obtained, at least in principle, for specific classes of semigroups is somewhat obvious and, in any case, will be explained later, in Section 4.

Let S be a nonempty semigroup and T a sequence of terms from S . We call T (weakly) **idempotent-product free** if T contains no nonempty subsequence the product whose terms, in any order, is an idempotent, and we call T **strongly idempotent-product free** if T contains no nonempty subsequence the product whose terms, in their natural order in T , is an idempotent.

In fact, by using almost the same idea of arguments employed by Gillam, Hall and Williams [8], we can derive the following proposition for any semigroup S such that $|S \setminus E(S)|$ is finite. For the readers' convenience, we shall give the arguments in Section 3.

Proposition 1.1. *Let S be a nonempty semigroup such that $|S \setminus E(S)|$ is finite. Then any S -valued sequence of length $|S \setminus E(S)| + 1$ is not strongly idempotent-product free.*

So, a natural question arises:

If S is a nonempty semigroup such that $|S \setminus E(S)|$ is finite, and T is a weakly (respectively, strongly) idempotent-product free S -valued sequence of length $|S \setminus E(S)|$, what can we say about T and the structure of S ?

In this manuscript, we completely answered this question in case that T is a weakly idempotent-product free S -valued sequence of length $|S \setminus E(S)|$. For the sake of exposition, we shall present the main theorem together with its proof in Section 3. Section 2 contains only some necessary preliminaries. In the final Section 4, further researches are proposed.

2 Some Preliminaries

We begin by recalling some notations extensively used in zero-sum theory, though mostly in the setting of commutative groups, see ([6], Chapter 5) for abelian groups and see [11] for nonabelian groups.

Let S be a nonempty semigroup. Finite S -valued sequences can be regarded as words in the free monoid $\mathcal{F}(S)$ with basis S , we denote them multiplicatively, so as to write $x_1 x_2 \cdots x_\ell$ in

place of $(x_1, x_2, \dots, x_\ell)$, and call them simply sequences. We say the sequence $T = x_1 x_2 \cdots x_\ell \in \mathcal{F}(\mathcal{S})$ has length $|T| = \ell$. We say $T' = x_{i_1} x_{i_2} \cdots x_{i_t}$ is a subsequence of T provided that $t \in [0, \ell]$ and $1 \leq i_1 < i_2 < \dots < i_t \leq \ell$. Note that the operation (connecting two sequences) of $\mathcal{F}(\mathcal{S})$ is represented by \cdot , which is different from the operation of \mathcal{S} . Accordingly, we write x^n for the n -fold product of an element $x \in \mathcal{S}$, and $T^{[n]}$ for the n -fold product of the sequence $T \in \mathcal{F}(\mathcal{S})$. By $TT'^{[-1]}$ we denote the remaining subsequence of T obtained by deleting the terms of T' from T . For any element $x \in \mathcal{S}$, by $v_x(T)$ we denote the multiplicity of x in the sequence T , i.e., the times which x appears to be terms in the sequence T . We set $\text{supp}(T) = \{x \in \mathcal{S} : v_x(T) > 0\}$. Let σ be any permutation of $\{1, 2, \dots, \ell\}$. By $\pi_\sigma(T)$ we denote the product $x_{\sigma(1)} * x_{\sigma(2)} * \cdots * x_{\sigma(\ell)}$ of terms of T in the order under the permutation σ . If σ is the identity permutation, we just write $\pi(T)$ simply for $\pi_\sigma(T)$. Let

$$\prod(T) = \{\pi_\sigma(T') : T' \text{ takes every nonempty subsequence of } T \text{ and } \sigma \text{ takes every permutation of } [1, |T'|]\}.$$

We call T a (weakly) *idempotent-product free* \mathcal{S} -valued sequence by meaning that

$$\prod(T) \cap E(\mathcal{S}) = \emptyset,$$

and T a *strongly idempotent-product free* \mathcal{S} -valued sequence by meaning that

$$\{\pi(T') : T' \text{ takes every nonempty subsequence of } T\} \cap E(\mathcal{S}) = \emptyset.$$

For any element x of \mathcal{S} , we define

$$\lambda_T(x) = |\prod(T \cdot x) \setminus \prod(T)|.$$

The zero element of \mathcal{S} , denoted $0_{\mathcal{S}}$ (if it exists), is the unique element z of \mathcal{S} such that $z * x = x * z = z$ for every $x \in \mathcal{S}$.

Let X be a subset of \mathcal{S} . We say X generates \mathcal{S} , or the elements of X are generators of \mathcal{S} , provided that every element $s \in \mathcal{S}$ is the product of one or more elements in X , in which case we write $\mathcal{S} = \langle X \rangle$. In particular, we use $\langle x \rangle$ in place of $\langle X \rangle$ if $X = \{x\}$, and we say that \mathcal{S} is a cyclic semigroup if it is generated by a single element. For any element $x \in \mathcal{S}$ such that $\langle x \rangle$ is finite, the least integer $r > 0$ such that $x^r = x^t$ for some positive integer $t \neq r$ is the **index** of x , denoted $I(x)$, then the least integer $k > 0$ such that $x^{I(x)+k} = x^{I(x)}$ is the **period** of x , denoted $\mathcal{P}(x)$. Let I be an ideal of the semigroup \mathcal{S} , the relation defined by

$$a \mathcal{I} b \Leftrightarrow a = b \text{ or } a, b \in I$$

is a congruence on \mathcal{S} , the Rees Congruence of the ideal I . The quotient semigroup $\mathcal{S}/I = \mathcal{S}/\mathcal{I}$ is the Rees quotient of \mathcal{S} by I . Let Q be a semigroup with zero disjoint from \mathcal{S} . An **ideal extension** of \mathcal{S} by Q is a semigroup B such that \mathcal{S} is an ideal of B and the Rees quotient

$B/S = Q$. A **partial homomorphism** of $Q^* = Q \setminus \{0_Q\}$ into a semigroup \mathcal{D} is a mapping $f : Q^* \rightarrow \mathcal{D}$ such that $f(a * b) = f(a) * f(b)$ whenever $a * b \neq 0_Q$.

If \mathcal{S} is a commutative semigroup, it is then possible to define a fundamental congruence, $\mathcal{N}_\mathcal{S}$, on \mathcal{S} as follows: Let a, b be any two elements of \mathcal{S} . We write $a \leq_{\mathcal{N}_\mathcal{S}} b$ to mean that $a^m = b * c$ for some $c \in \mathcal{S}$ and some integer $m > 0$. If $a \leq_{\mathcal{N}_\mathcal{S}} b$ and $b \leq_{\mathcal{N}_\mathcal{S}} a$, we write $a \mathcal{N}_\mathcal{S} b$. We call the commutative semigroup \mathcal{S} an archimedean semigroup provided that $a \mathcal{N}_\mathcal{S} b$ for any two elements a, b of \mathcal{S} . By ([10], Chapter III, Theorem 1.2), the quotient semigroup $Y(\mathcal{S}) = \mathcal{S}/\mathcal{N}_\mathcal{S}$ is a lower semilattice, called the **universal semilattice** of \mathcal{S} . Furthermore, there exists a partition $\mathcal{S} = \bigcup_{y \in Y(\mathcal{S})} \mathcal{S}_y$ into subsemigroups \mathcal{S}_y (one for every $y \in Y(\mathcal{S})$) with respect to the universal semilattice $Y(\mathcal{S})$, in particular, $\mathcal{S}_{y_1} * \mathcal{S}_{y_2} \subseteq \mathcal{S}_{y_1 \wedge y_2}$ for all $y_1, y_2 \in Y(\mathcal{S})$, and each component \mathcal{S}_y is archimedean. The following lemma to characterize the structure of any finite commutative archimedean semigroup will be useful for the proof later.

Lemma 2.1. ([10], Chapter I, Proposition 3.6, Proposition 3.7, Proposition 3.8, and Chapter III, Proposition 3.1) *A finite commutative semigroup \mathcal{S} is archimedean if and only if it is an ideal extension of a finite abelian group G by a finite commutative nilsemigroup N . Moreover, the partial homomorphism $\varphi_G^N : N \setminus \{0_N\} \rightarrow G$ to construct the ideal extension of the group G by the nilsemigroup N is given by*

$$\varphi_G^N : a \mapsto a * e_G$$

where a denotes an arbitrary element $N \setminus \{0_N\} = \mathcal{S} \setminus G$ and e_G denotes the identity element of the subgroup G .

We say that the semigroup \mathcal{S} is a nilsemigroup if every element of \mathcal{S} is nilpotent, i.e., \mathcal{S} has a zero element $0_\mathcal{S}$ and for each element $x \in \mathcal{S}$ there exists an integer $n > 0$ such that $x^n = 0_\mathcal{S}$.

The following lemmas will be useful for our arguments.

Lemma 2.2. (see [9], Chapter IV, p. 127) *Let N be a finite commutative nilsemigroup, and let a, b be two elements of N . If $a * b \in \{a, b\}$, then $a = 0_N$ or $b = 0_N$.*

Lemma 2.3. ([10], Chapter I, Lemma 5.7, Proposition 5.8, Corollary 5.9) *Let $\mathcal{S} = \langle x \rangle$ be a finite cyclic semigroup. Then $\mathcal{S} = \{x, x^2, \dots, x^{I(x)}, x^{I(x)+1}, \dots, x^{I(x)+\mathcal{P}(x)-1}\}$ with*

$$x^i * x^j = \begin{cases} x^{i+j}, & \text{if } i + j \leq I(x) + \mathcal{P}(x) - 1; \\ x^k, & \text{if } i + j \geq I(x) + \mathcal{P}(x), \text{ where} \\ & I(x) \leq k \leq I(x) + \mathcal{P}(x) - 1 \text{ and } k \equiv i + j \pmod{\mathcal{P}(x)}. \end{cases}$$

Moreover,

(i) *there exists a unique idempotent, x^ℓ , in the cyclic semigroup $\langle x \rangle$, where*

$$\ell \in [I(x), I(x) + \mathcal{P}(x) - 1] \text{ and } \ell \equiv 0 \pmod{\mathcal{P}(x)};$$

(ii) *$\{x^{I(x)}, x^{I(x)+1}, \dots, x^{I(x)+\mathcal{P}(x)-1}\}$ is a cyclic subgroup of \mathcal{S} isomorphic to the additive group $\mathbb{Z}_{\mathcal{P}(x)}$ of integers modulo $\mathcal{P}(x)$.*

3 The structure of the extremal sequence

In this section, we shall determine the structure of idempotent-product free \mathcal{S} -value sequences of length $|\mathcal{S} \setminus E(\mathcal{S})|$. The following lemma will be useful.

Lemma 3.1. *Let \mathcal{S} be a nonempty semigroup. Let T be an \mathcal{S} -valued sequence with $\prod(T) \cap E(\mathcal{S}) = \emptyset$, and let x be a term of T . Then $\lambda_{T, x^{[-1]}}(x) \geq 1$.*

Proof. Since $|\prod(T)|$ is finite, combined with Lemma 2.3 (i), we derive that $\langle x \rangle \not\subseteq \prod(T)$ no matter whether $\langle x \rangle$ is finite or infinite, and thus, $\langle x \rangle \not\subseteq \prod(Tx^{[-1]})$. Let k be the least positive integer such that $x^k \notin \prod(Tx^{[-1]})$. If $k = 1$, i.e., $x \notin \prod(Tx^{[-1]})$, then $x \in \prod(T) \setminus \prod(Tx^{[-1]})$ which implies $\lambda_{T, x^{[-1]}}(x) \geq 1$, done. Hence, we assume $k > 1$. Then $x^{k-1} \in \prod(Tx^{[-1]})$, and thus, $x^k = x^{k-1} * x \in \prod(Tx^{[-1]}) * x \subseteq \prod(T)$, which implies $\lambda_{T, x^{[-1]}}(x) \geq 1$. This completes the proof. \square

Proof of Proposition 1.1 Let $T = a_1 a_2 \cdots a_\ell \in \mathcal{F}(\mathcal{S})$ with length $\ell = |\mathcal{S} \setminus E(\mathcal{S})| + 1$, where $a_i \notin E(\mathcal{S})$. Suppose to the contrary that T is strongly idempotent-product free. Let

$$A_k = \{\pi(T_k) : T_k \text{ is a nonempty subsequence of } a_1 a_2 \cdots a_k\}$$

where $k \in [1, \ell]$. Clearly,

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\ell. \quad (1)$$

We shall prove that

$$|A_{t+1}| > |A_t| \text{ for each } t \in [1, \ell - 1]. \quad (2)$$

Since $|\mathcal{S} \setminus E(\mathcal{S})|$ is finite, we have that the cyclic subsemigroup $\langle a_{t+1} \rangle$ is finite and contains an idempotent. Let m be the least positive integer such that $a_{t+1}^m \notin A_t$. If $m = 1$ then $a_{t+1} \in A_{t+1} \setminus A_t$, and if $m > 1$ then $a_{t+1}^m = a_{t+1}^{m-1} * a_{t+1} \in A_{t+1} \setminus A_t$, which implies (2).

By (1) and (2), we conclude that $|A_\ell| \geq |A_1| + \ell - 1 = \ell = |\mathcal{S} \setminus E(\mathcal{S})| + 1$, a contradiction with T being strongly idempotent-product free. \square

Now we are in a position to give the main theorem.

Theorem 3.2. *Let \mathcal{S} be a nonempty semigroup such that $|\mathcal{S} \setminus E(\mathcal{S})|$ is finite, and let T be an \mathcal{S} -valued sequence of length $|\mathcal{S} \setminus E(\mathcal{S})|$. Then $\prod(T) \cap E(\mathcal{S}) = \emptyset$ if, and only if, $\mathcal{R} = \langle \text{supp}(T) \rangle$ is a finite commutative semigroup with $\mathcal{S} \setminus \mathcal{R} \subseteq E(\mathcal{S})$ and the universal semilattice $Y(\mathcal{R})$ is a chain such that $x_1 * x_2 = x_1$ for any elements $x_1, x_2 \in \mathcal{R}$ with $x_1 \not\leq_{N_{\mathcal{R}}} x_2$, and moreover,*

(i) *each archimedean component of \mathcal{R} is, either a finite cyclic semigroup $\langle x \rangle$ with $x \in \text{supp}(T)$ and $I(x) \equiv 1 \pmod{\mathcal{P}(x)}$, or an ideal extension of a nontrivial finite cyclic group $\langle x_2 \rangle$ by a nontrivial finite cyclic nilsemigroup $\langle x_1 \rangle$ with $x_1, x_2 \in \text{supp}(T)$ and the partial homomorphism*

$\varphi_{\langle x_2 \rangle}^{\langle x_1 \rangle}$ being trivial, i.e., $\varphi_{\langle x_2 \rangle}^{\langle x_1 \rangle}(x_1) = e_{\langle x_2 \rangle}$ where $e_{\langle x_2 \rangle}$ denotes the identity element of the subgroup $\langle x_2 \rangle$;

(ii) $v_x(T) = I(x) + \mathcal{P}(x) - 2$ for each element $x \in \text{supp}(T)$.

Proof of Theorem 3.2. The sufficiency is easy to verify. We need only to consider the necessity. Note first that the cyclic semigroup $\langle a \rangle$ is finite for every non-idempotent element $a \in \mathcal{S}$, since otherwise, $\langle a \rangle$ would be isomorphic to the additive semigroup \mathbb{N}^+ , which is a contradiction with $|\mathcal{S} \setminus E(\mathcal{S})|$ being finite. Let $\ell = |T| = |\mathcal{S} \setminus E(\mathcal{S})|$ and $T = a_1 a_2 \cdots a_\ell \in \mathcal{F}(\mathcal{S})$ with $\prod(T) \cap E(\mathcal{S}) = \emptyset$. Let τ denote an arbitrary permutation of $\{1, 2, \dots, \ell\}$, and let

$$T_k^\tau = a_{\tau(1)} a_{\tau(2)} \cdots a_{\tau(k)}$$

for each $k \in [1, m]$. Since $\prod(T_k^\tau) \cap E(\mathcal{S}) = \emptyset$ for all $k \in [1, \ell]$, it follows from Lemma 3.1 that

$$\begin{aligned} |T| &= |\mathcal{S} \setminus E(\mathcal{S})| \\ &\geq |\prod(T)| = |\prod(T_{\ell-1}^\tau)| + \lambda_{T_{\ell-1}^\tau}(a_{\tau(\ell)}) \\ &\geq |\prod(T_{\ell-1}^\tau)| + 1 = |\prod(T_{\ell-2}^\tau)| + \lambda_{T_{\ell-2}^\tau}(a_{\tau(\ell-1)}) + 1 \\ &\geq |\prod(T_{\ell-2}^\tau)| + 2 \\ &\vdots \\ &\geq |\prod(T_1^\tau)| + \ell - 1 = \ell = |T|. \end{aligned}$$

It follows that

$$|\prod(T_k^\tau)| = k \tag{3}$$

for each $k \in [1, \ell]$, and that

$$\prod(T) = \mathcal{S} \setminus E(\mathcal{S}). \tag{4}$$

Then we have the following.

Claim A. If a, b are two distinct elements of $\text{supp}(T)$, then $a * b = b * a \in \{a, b\}$.

Proof of Claim A. By (3) and the arbitrariness of τ , we have that $|\prod(a \cdot b)| = 2$, which implies $a * b, b * a \in \{a, b\}$. Suppose to the contrary without loss of generality that $a * b \neq b * a$ with $a * b = b$ and $b * a = a$. It follows that $a * a = a * (b * a) = (a * b) * a = b * a = a$, and so a is an idempotent, which is absurd. This proves Claim A. \square

By Claim A, then $\mathcal{R} = \langle \text{supp}(T) \rangle$ is **commutative**. Moreover, we have the following.

Claim B.

$$\mathcal{R} = \bigcup_{a \in \text{supp}(T)} \langle a \rangle.$$

In particular, for any $x \in \prod(T)$, there exists an element $a \in \text{supp}(T)$ such that $x = a^k$ with $k \in [1, v_a(T)]$.

Proof of Claim B. Take an arbitrary element x of \mathcal{R} . There exists some distinct elements of $\text{supp}(T)$, say x_1, x_2, \dots, x_m , such that $x = x_1^{n_1} * x_2^{n_2} * \dots * x_m^{n_m}$, where $m > 0$ and $n_1, n_2, \dots, n_m > 0$. By applying Claim A, we conclude that $x = x_i^{n_i}$ for some $i \in [1, m]$. In particular, if $x \in \prod(T)$, we can take all the integers n_1, n_2, \dots, n_m above such that $n_i \in [1, v_{x_i}(T)]$ for every $i \in \{1, 2, \dots, m\}$. This proves Claim B. \square

By Claim B, we see that \mathcal{R} is **finite** and we have the following.

Claim C. For any $a \in \text{supp}(T)$ and any integer $k \in [1, I(a) + \mathcal{P}(a) - 1]$ such that $a^k \in \prod(T)$,

$$v_a(T) \geq k.$$

Proof of Claim C. By Claim B, we have that $a^k = b^t$ for some $b \in \text{supp}(T)$ with $t \in [1, v_b(T)]$. Suppose $b \neq a$. It follows from Claim A that $a^k * a^k = a^k * b^t = b^t = a^k$, and thus a^k is an idempotent, a contradiction. Hence, $b = a$ and $v_a(T) = v_b(T) \geq t \geq k$. This proves Claim C. \square

Let g and h be two arbitrary elements of \mathcal{R} which belong to two distinct archimedean components of \mathcal{R} . By Claim B, we have $g = a^k$ and $h = b^t$ where a, b are distinct elements of $\text{supp}(T)$ and $k, t > 0$. It follows from Claim A that

$$g * h = a^k * b^t = a^k = g$$

or

$$g * h = a^k * b^t = b^t = h$$

which implies

$$g \preceq_{\mathcal{N}_{\mathcal{R}}} h$$

or

$$h \preceq_{\mathcal{N}_{\mathcal{R}}} g.$$

Since $\mathcal{N}_{\mathcal{R}}$ is a congruence on \mathcal{R} , by the arbitrariness of g and h , we conclude that the universal semilattice $Y(\mathcal{R}) = \mathcal{R}/\mathcal{N}_{\mathcal{R}}$ is a chain and $g * h = g$ for any elements $g, h \in \mathcal{R}$ with $g \preceq_{\mathcal{N}_{\mathcal{R}}} h$.

Let a be an arbitrary element of $\text{supp}(T)$. By (4), we have that all the elements except for the unique idempotent of $\langle a \rangle$ must belong to $\prod(T)$. Combined with Lemma 2.3 and Claim C, we conclude that

$$v_a(T) = I(a) + \mathcal{P}(a) - 2, \tag{5}$$

and that the unique idempotent in the cyclic semigroup $\langle a \rangle$ is $a^{I(a)+\mathcal{P}(a)-1}$ which implies $I(a) + \mathcal{P}(a) - 1 \equiv 0 \pmod{\mathcal{P}(a)}$, equivalently,

$$I(a) \equiv 1 \pmod{\mathcal{P}(a)}. \tag{6}$$

By (5), we have Conclusion (ii) proved. Now it remains to show Conclusion (i).

Let A_y ($y \in Y(\mathcal{R})$) be an arbitrary archimedean component of \mathcal{R} . Since $x \mathcal{N}_{\mathcal{R}} x^t$ for any element $x \in \mathcal{R}$ and any integer $t > 0$, we conclude by Claim B that A_y is a union of several cyclic subsemigroups generated by the elements of $\text{supp}(T)$, i.e.,

$$A_y = \bigcup_{i=1}^{k_y} \langle x_i \rangle, \quad (7)$$

where $k_y \geq 1$ and x_1, x_2, \dots, x_{k_y} are distinct elements of $\text{supp}(T)$. By Lemma 2.1, we may assume that A_y is an ideal extension of a group G_y by a nilsemigroup N_y (note that G_y or N_y may be trivial which shall be reduced to the case that A_y is a nilsemigroup or a group). Now we show that

$$|G_y \cap \text{supp}(T)| \leq 1 \quad (8)$$

and

$$|(A_y \setminus G_y) \cap \text{supp}(T)| \leq 1. \quad (9)$$

Suppose a, b are two distinct elements of $A_y \cap \text{supp}(T)$. Recalling Claim A, we see

$$a * b \in \{a, b\}.$$

If $a, b \in G_y$, then a or b is the identity element of the group G_y which is an idempotent, a contradiction. If $a, b \in A_y \setminus G_y = N_y \setminus \{0_{N_y}\}$, by Lemma 2.2, we derive a contradiction. This proves (8) and (9).

By (8) and (9), we have that

$$k_y \in \{1, 2\}$$

in (7).

Consider the case of $k_y = 1$, i.e., $A_y = \langle x \rangle$ for some $x \in \text{supp}(T)$. Combined with (6), we have Conclusion (i) proved.

Consider the case of $k_y = 2$, i.e., $A_y = \langle x_1 \rangle \cup \langle x_2 \rangle$ where x_1 and x_2 are distinct elements of $\text{supp}(T)$. By (8) and (9), we may assume without loss of generality that $x_2 \in G_y$ and $x_1 \in A_y \setminus G_y = N_y \setminus \{0_{N_y}\}$. Combined with Claim A, we see $x_1 * x_2 = x_2$. Then we conclude that the partial homomorphism $\varphi_{\langle x_2 \rangle}^{\langle x_1 \rangle}$ is trivial, and $G_y = \langle x_2 \rangle$ and $N_y = \langle x_1 \rangle$, and so Conclusion (i) holds.

This completes the proof of Theorem 3.2. □

It is not hard to see that Theorem 3.2 can be also stated as the following equivalent form.

Let \mathcal{S} be a nonempty semigroup such that $|\mathcal{S} \setminus E(\mathcal{S})|$ is finite, and let T be an \mathcal{S} -valued sequence of length $|\mathcal{S} \setminus E(\mathcal{S})|$. Then $\prod(T) \cap E(\mathcal{S}) = \emptyset$ if, and only if, $\mathcal{R} = \langle \text{supp}(T) \rangle$ is a finite commutative semigroup such that $\mathcal{S} \setminus \mathcal{R} \subseteq E(\mathcal{S})$ and

$$\mathcal{R} = \bigcup_{i=1}^k \langle x_i \rangle$$

where $\text{supp}(T) = \{x_1, x_2, \dots, x_k\}$, $x_i * x_j = x_j$ and $\langle x_i \rangle^\circ \cap \langle x_j \rangle^\circ = \emptyset$ for all $1 \leq i < j \leq k$, and $\langle x \rangle^\circ$ denotes the subset of all non-idempotent elements in the finite cyclic semigroup $\langle x \rangle^\circ$, and moreover, $I(x_i) \equiv 1 \pmod{\mathcal{P}(x_i)}$ and $v_{x_i}(T) = I(x_i) + \mathcal{P}(x_i) - 2$ for every $i \in \{1, 2, \dots, k\}$.

4 Concluding remarks

We remark that the value $|\mathcal{S} \setminus E(\mathcal{S})| + 1$ is best possible to ensure that any \mathcal{S} -valued sequence of length $|\mathcal{S} \setminus E(\mathcal{S})| + 1$ is not (strongly) idempotent-product free, in the sense that \mathcal{S} is a general semigroup. However, this value may be no longer best possible for a particular kind of semigroups. Hence, we introduce the following two combinatorial constants for any semigroup \mathcal{S} .

Definition 4.1. *Let \mathcal{S} be a nonempty semigroup (not necessarily finite). We define $I(\mathcal{S})$, which is called the **Erdős-Burgess constant** of the semigroup \mathcal{S} , to be the least $\ell \in \mathbb{N} \cup \{\infty\}$ such that every \mathcal{S} -valued sequence T of length ℓ is not (weakly) idempotent-product free, and we define $SI(\mathcal{S})$, which is called the **strong Erdős-Burgess constant** of the semigroup \mathcal{S} , to be the least $\ell \in \mathbb{N} \cup \{\infty\}$ such that every \mathcal{S} -valued sequence of length ℓ is not strongly idempotent-product free. Formally, we can also define*

$$I(\mathcal{S}) = \sup \{|T| + 1 : T \text{ takes every idempotent-product free } \mathcal{S}\text{-valued sequence}\}$$

and

$$SI(\mathcal{S}) = \sup \{|T| + 1 : T \text{ takes every strongly idempotent-product free } \mathcal{S}\text{-valued sequence}\}.$$

Then we have the following.

Proposition 4.2. *Let \mathcal{S} be a nonempty semigroup.*

- (i). *If $I(\mathcal{S})$ or $SI(\mathcal{S})$ is finite then $\langle x \rangle$ is finite for every element $x \in \mathcal{S}$;*
- (ii). *$I(\mathcal{S}) \leq SI(\mathcal{S})$, and if \mathcal{S} is commutative then $I(\mathcal{S}) = SI(\mathcal{S})$; In particular, for the case $|\mathcal{S} \setminus E(\mathcal{S})|$ is finite, $I(\mathcal{S}) = |\mathcal{S} \setminus E(\mathcal{S})| + 1$ holds if, and only if, the semigroup \mathcal{S} is given as in Theorem 3.2.*

Proof. Conclusion (ii) follows from the definition and Theorem 3.2 readily.

- (i). Suppose to the contrary the there exists some element $x \in \mathcal{S}$ such that $\langle x \rangle$ is infinite. Then the semigroup $\langle x \rangle$ is isomorphic the additive semigroup \mathbb{N}^+ . The idempotent-product free sequence $x^{[\ell]}$ of arbitrarily great length $\ell \in \mathbb{N}$ gives the contradiction. \square

The prerequisite that $\langle x \rangle$ is finite for every element $x \in S$, is necessary for $I(S)$ ($SI(S)$) being finite but not sufficient. For example, take a semigroup

$$S = \langle \{x_i : i \in \mathbb{N}\} \rangle \quad (10)$$

where $x_i * x_j = x_j * x_i = x_j$ for any $1 \leq i < j$, and where $\langle x_t \rangle$ is a finite cyclic group of order $t + 1$ for $t \in \mathbb{N}$. It is not hard to check that $x_1 x_2 \cdots x_k$ is an idempotent-product free S -valued sequence of length k for any $k \in \mathbb{N}$, which gives that the infinity of $I(S)$ ($SI(S)$).

Hence, the following problems would be interesting.

Problem 1. *Let S be a nonempty semigroup. Does there exist sufficient and necessary conditions to decide whether $I(S)$ ($SI(S)$) is finite or not?*

Problem 2. *Let S be a nonempty semigroup. Does there exist sufficient and necessary conditions to decide whether $I(S) = SI(S)$ or not?*

One thing worth remarking is that $I(S)$ is finite does not imply that $SI(S)$ is finite. For example, take the semigroup $S = \langle \{x_i : i \in \mathbb{N}\} \rangle \cup \{0_S\}$ with zero element where $x_i * x_j = x_j$ and $x_j * x_i = 0_S$ for any $1 \leq i < j$, and where $\langle x_t \rangle$ is a finite cyclic group of order some fixed integer $m > 2$ for all $t \in \mathbb{N}$. It is easy to check that $I(S) = m$ and $SI(S)$ is infinite.

Problem 3. *Let S be a nonempty semigroup such that $|S \setminus E(S)|$ is finite. Find the sufficient and necessary conditions to decide whether $SI(S) = |S \setminus E(S)| + 1$. Moreover, in case that $SI(S) = |S \setminus E(S)| + 1$, determine the structure of the strongly idempotent-product free S -valued sequences of length $|S \setminus E(S)|$.*

We remark that the above Problem 3 is in fact the inverse problem of Gillam, Hall and Williams (see Proposition 1.1).

Problem 4. *For some important kind of semigroup S , determine the values of $I(S)$ and $SI(S)$.*

In the case that the semigroup S is commutative, the (strong variant is the same as shown in Proposition 4.2) Erdős-Burgess constant seems to be closely related to a classical combinatorial constant, the **Davenport constant** originated from K. Rogers [13]. Davenport constant is the most important constant in Zero-sum Theory which has been extensively investigated for abelian groups since the 1960s (see [3–5, 7, 12]), and recently was also studied for commutative semigroups (see [1, 14–19], and P. 110 in [6]). For the readers' convenience, we state the definition of Davenport constant for commutative semigroups below.

Definition 4.3. ([14–16]) *Let S be a commutative semigroup. Define $D(S)$ to be the least $\ell \in \mathbb{N} \cup \{\infty\}$ such that every S -valued sequence T of length at least ℓ contains a proper subsequence T' ($T' \neq T$) the product whose terms is equal to the product of all terms in T .*

It is easy to see that for the case that S is an abelian group, both constants really mean the same thing, i.e., $I(S) = D(S)$. While, for the case that the commutative semigroup S is

not a group, both $I(S) < D(S)$ and $I(S) > D(S)$ could happen, which can be noticed from the following example.

Example. Take a commutative semigroup $S = \langle x_1 \rangle \cup \langle x_2 \rangle$ where $\langle x_1 \rangle$ is a finite cyclic group and $\langle x_2 \rangle$ is a finite cyclic nilsemigroup with $x_1 * x_2 = x_2 * x_1 = x_2$ and $|\langle x_1 \rangle| = n_1$ and $|\langle x_2 \rangle| = n_2$. Then we check that $I(S) = (n_1 - 1) + (n_2 - 1) + 1$ and $D(S) = \max(n_1, n_2 + 1)$. By taking proper n_1, n_2 , we have that both $I(S) < D(S)$ and $I(S) > D(S)$ could happen.

Therefore, we close this manuscript by proposing the following problem.

Problem 5. Let S be a commutative semigroup. Does there exist any relationship between the Erdős-Burgess constant $I(S)$ and the Davenport constant $D(S)$?

Acknowledgements

This work is supported by NSFC (11301381, 11271207), Science and Technology Development Fund of Tianjin Higher Institutions (20121003).

References

- [1] S.D. Adhikari, W. Gao and G. Wang, Erdős-Ginzburg-Ziv theorem for finite commutative semigroups, *Semigroup Forum*, **88** (2014) 555–568.
- [2] D.A. Burgess, A problem on semi-groups, *Studia Sci. Math. Hungar.*, **4** (1969) 9–11.
- [3] P. van Emde Boas and D. Kruyswijk, A combinatorial problem on finite abelian groups, 3, Report ZW 1969-008, Stichting Math. Centrum, Amsterdam.
- [4] W. Gao, On Davenport's constant of finite abelian groups with rank three, *Discrete Math.*, **222** (2000) 111–124.
- [5] A. Geroldinger, Additive Group Theory and Non-unique Factorizations, 1–86 in: A. Geroldinger and I. Ruzsa (Eds.), *Combinatorial Number Theory and Additive Group Theory* (Advanced Courses in Mathematics-CRM Barcelona), Birkhäuser, Basel, 2009.
- [6] A. Geroldinger and F. Halter-Koch, Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory, *Pure and Applied Mathematics*, vol. 278, Chapman & Hall/CRC, 2006.
- [7] A. Geroldinger and R. Schneider, On Davenport's constant, *J. Combin. Theory Ser. A*, **61** (1992) 147–152.
- [8] D.W.H. Gillam, T.E. Hall and N.H. Williams, On finite semigroups and idempotents, *Bull. Lond. Math. Soc.*, **4** (1972) 143–144.

- [9] P.A. Grillet, *Semigroups: An introduction to the Structure Theory*, Dekker, New York, 1995.
- [10] P.A. Grillet, *Commutative Semigroups*, Kluwer Academic Publishers, 2001.
- [11] D.J. Gryniewicz, The large Davenport constant II: General upper bounds, *J. Pure Appl. Algebra*, **217** (2013) 2221–2246.
- [12] J.E. Olson, A Combinatorial Problem on Finite Abelian Groups, I, *J. Number Theory*, **1** (1969) 8–10.
- [13] K. Rogers, A Combinatorial problem in Abelian groups, *Math. Proc. Cambridge Philos. Soc.*, **59** (1963) 559–562.
- [14] G. Wang, Davenport constant for semigroups II, *J. Number Theory*, **153** (2015) 124–134.
- [15] G. Wang, Additively irreducible sequences in commutative semigroups, *arXiv:1504.06818*.
- [16] G. Wang and W. Gao, Davenport constant for semigroups, *Semigroup Forum*, **76** (2008) 234–238.
- [17] G. Wang and W. Gao, Davenport constant of the multiplicative semigroup of the ring $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, *arXiv:1603.06030*.
- [18] H.L. Wang, L.Z. Zhang, Q.H. Wang and Y.K. Qu, Davenport constant of the multiplicative semigroup of the quotient ring $\frac{\mathbb{F}_p[x]}{\langle f(x) \rangle}$, *International Journal of Number Theory*, in press, DOI: 10.1142/S1793042116500433.
- [19] L.Z. Zhang, H.L. Wang and Y.K. Qu, A problem of Wang on Davenport constant for the multiplicative semigroup of the quotient ring of $\mathbb{F}_2[x]$, *arXiv:1507.03182*.